Eigenvalue and Dirichlet problem for fully-nonlinear operators in non smooth domains.

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1   Introduction

In this paper we study the maximum principle, the existence of eigenvalue and the existence of solution for the Dirichlet problem for operators which are fully-nonlinear, elliptic but presenting some singularity or degeneracy which are similar to those of the p-Laplacian, the novelty resides in the fact that we consider the equations in bounded domains which only satisfy the exterior cone condition.

Before defining the precise notions described above let us recall that Berestycki, Nirenberg and Varadhan in [1], have proved maximum principle, principal eigenvalue and Dirichlet problem for linear uniformly elliptic operators $Lu = \operatorname{tr} A(x)D^2 u + b(x) \cdot \nabla u + c(x)u$ in domains without any regularity condition on the boundary.

In order to do so, they need to define the concept of boundary condition. Hence, using Alexandrov Bakelman Pucci inequality and Krylov-Safonov Harnack's inequality they first prove the existence of u_o a strong solution of

$$\operatorname{tr} A(x)D^2 u_o + b(x) \cdot \nabla u_o = -1 \quad \text{in} \quad \Omega$$

which is zero on the points of the boundary that have some smoothness. Then they define the boundary condition for the full operator L through this function u_o. Their paper, which constructs the principal eigenvalue using only the maximum principle, has allowed to generalize the notion of eigenvalue to fully-nonlinear operators, see e.g. [6, 18, 10, 12, 3, 4, 16].

Here, as in [3, 4, 16] we shall consider operators that satisfy:

(H1) $F : \Omega \times \mathbb{R}^N \setminus \{0\} \times S \to \mathbb{R}$, and $\forall t \in \mathbb{R}^\star$, $\mu \geq 0$, $F(x, tp, \mu X) = |t|^\alpha \mu F(x, p, X)$.

(H2) There exist $0 < a < A$, for $x \in \overline{\Omega}$, $p \in \mathbb{R}^N \setminus \{0\}$, $M \in S$, $N \in S$, $N \geq 0$

$$a|p|^\alpha tr(N) \leq F(x, p, M + N) - F(x, p, M) \leq A|p|^\alpha tr(N).$$

and other "regularity" conditions.

For this class of operators it is not known whether the Alexandrov Bakelman Pucci inequality holds true, hence in our previous works [4, 3] we supposed that $\partial\Omega$ was $\mathcal{C}^2$. The regularity of the boundary in those papers played a crucial role because it allowed to use the distance function to construct sub and super solutions. This was the key step in the proof of the maximum principle. Here, instead, we shall suppose that Ω satisfies only the "uniform exterior" cone condition i.e.
There exist $\psi > 0$ and $\bar{r} > 0$ such that for any $z \in \partial\Omega$ and for an axe through z of direction $\vec{n}$,

$$Co := \{x : \frac{(x-z) \cdot \vec{n}}{|z - x|} \leq \cos\psi\}, \qquad Co \cap \overline{\Omega} \cap B_{\bar{r}}(z) = \{z\}.$$

This cone condition allows to construct some barriers and consequently a function which will play the same role as u_o in [1]. In particular we can prove that there exists an eigenfunction $\varphi > 0$, solution of

$$\begin{cases} F(x, \nabla\varphi, D^2\varphi) + h(x) \cdot \nabla\varphi|\nabla\varphi|^\alpha + (V(x) + \bar{\lambda}(\Omega))\varphi^{1+\alpha} = 0 & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega \end{cases}$$

for

$$\begin{aligned} \bar{\lambda}(\Omega) \;=\; & \sup\{\lambda, \exists\, u > 0 \text{ in } \Omega, \\ & F(x, \nabla u, D^2 u) + h(x) \cdot \nabla u|\nabla u|^\alpha + (V(x) + \lambda)u^{1+\alpha} \leq 0 \text{ in } \Omega\}. \end{aligned}$$

Finally in the last section we also define

$$\lambda_e = \sup\{\lambda(\Omega'), \; \Omega \subset\subset \Omega', \; \Omega' \text{ regular and bounded}\}$$

and

$$\begin{aligned} \tilde{\lambda} \;=\; & \sup\{\lambda, \exists\, u > 0 \text{ in } \overline{\Omega}, \\ & F(x, \nabla u, D^2 u) + h(x) \cdot \nabla u|\nabla u|^\alpha + (V(x) + \lambda)u^{1+\alpha} \leq 0\}. \end{aligned}$$

We prove that $\lambda_e = \tilde{\lambda}$ and that this value is an "eigenvalue" in the sense that there exists some $\phi_e > 0$, which satisfies

$$\begin{cases} F(x, \nabla\phi_e, D^2\phi_e) + h(x)\cdot\nabla\phi_e|\nabla\phi_e|^\alpha + (V(x) + \lambda_e(\Omega))\phi_e^{1+\alpha} = 0 & \text{in} \quad \Omega \\ \phi_e = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

We also prove that for any $\lambda < \lambda_e$ the maximum principle holds and there exists a solution of the Dirichlet problem when the right hand side is negative.

Observe that $\lambda_e \leq \overline{\lambda}$, and furthermore if Ω is smooth, the equality holds. It is an open problem to know if if the equality still holds when Ω satisfies only the exterior cone condition. Let us observe that the identity of these values is equivalent to the existence of a maximum principle for $\lambda < \overline{\lambda}$.

2 Assumptions on F

The following hypothesis will be considered

(H1) $F : \Omega \times \mathbb{R}^N \setminus \{0\} \times S \rightarrow \mathbb{R}$, and $\forall t \in \mathbb{R}^\star$, $\mu \geq 0$, $F(x, tp, \mu X) = |t|^\alpha \mu F(x, p, X)$.

(H2) There exist $0 < a < A$, for $x \in \overline{\Omega}$, $p \in \mathbb{R}^N\setminus\{0\}$, $M \in S$, $N \in S$, $N \geq 0$

$$a|p|^\alpha tr(N) \leq F(x, p, M + N) - F(x, p, M) \leq A|p|^\alpha tr(N). \qquad (2.1)$$

(H3) There exists a continuous function $\tilde{\omega}$, $\tilde{\omega}(0) = 0$ such that for all $(x, y) \in \Omega^2$, $\forall p \neq 0$, $\forall X \in S$

$$|F(x, p, X) - F(y, p, X)| \leq \tilde{\omega}(|x - y|)|p|^\alpha|X|.$$

(H4) There exists a continuous function ω with $\omega(0) = 0$, such that if $(X, Y) \in S^2$ and $\zeta \in \mathbb{R}^+$ satisfy

$$-\zeta \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq 4\zeta \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

and I is the identity matrix in $\mathbb{R}^N$, then for all $(x, y) \in \mathbb{R}^N$, $x \neq y$

$$F(x, \zeta(x - y), X) - F(y, \zeta(x - y), -Y) \leq \omega(\zeta|x - y|^2).$$

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Remark 2.1 *When no ambiguity arises we shall sometime write $F[u]$ to signify $F(x, \nabla u, D^2 u)$.*

We assume that h and V are some continuous and bounded functions on $\overline{\Omega}$ and

(H5) - Either $\alpha \leq 0$ and h is Hölder continuous of exponent $1 + \alpha$,
    - or $\alpha > 0$ and $h(x) - h(y) \cdot x - y \leq 0$

The solutions that we consider will be taken in the sense of viscosity. For convenience of the reader we state the precise definition.

Definition 2.2 *Let Ω be a bounded domain in $\mathbb{R}^N$, then v, bounded and continuous on $\overline{\Omega}$ is called a viscosity super solution (respectively sub-solution) of $F(x, \nabla u, D^2 u) + h(x) \cdot \nabla u |\nabla u|^\alpha = f(x, u)$ if for all $x_0 \in \Omega$,*
    -Either there exists an open ball $B(x_0, \delta)$, $\delta > 0$ in Ω on which $v = cte = c$ and $0 \leq f(x, c)$, for all $x \in B(x_0, \delta)$ (respectively $0 \geq f(x, c)$)
    -Or $\forall \varphi \in \mathcal{C}^2(\Omega)$, such that $v - \varphi$ has a local minimum on x_0 (respectively a local maximum) and $\nabla \varphi(x_0) \neq 0$, one has

$$F(x_0, \nabla\varphi(x_0), D^2\varphi(x_0)) + h(x_0) \cdot \nabla\varphi(x_0)|\nabla\varphi(x_0)|^\alpha \leq f(x_0, v(x_0)).$$

(respectively

$$F(x_0, \nabla\varphi(x_0), D^2\varphi(x_0)) + h(x_0) \cdot \nabla\varphi(x_0)|\nabla\varphi(x_0)|^\alpha \geq f(x_0, v(x_0)).)$$

We now recall what we mean by first eigenvalue and some of the properties of this eigenvalue.
    When Ω is a bounded domain we define

$$\bar{\lambda}(\Omega) = \sup\{\lambda, \exists \ \varphi > 0 \text{ in } \Omega, F[\varphi] + h(x) \cdot \nabla\varphi|\nabla\varphi|^\alpha + (V(x) + \lambda)\varphi^{1+\alpha} \leq 0\}$$

When Ω is a bounded regular set, we proved in [2] that

Theorem 2.3 *Suppose that F satisfies (H1)-(H4), that h satisfies (H5), and that V is continuous and bounded. Suppose that Ω is a bounded regular domain.*
    There exists $\varphi > 0$ which is a solution of

$$\begin{cases} F[\varphi] + h(x) \cdot \nabla\varphi|\nabla\varphi|^\alpha + (V(x) + \bar{\lambda})\varphi^{1+\alpha} = 0 & \text{in } \Omega \\ \qquad\qquad\qquad \varphi = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover φ is strictly positive inside Ω and is Hölder continuous

We now recall some properties of the eigenvalue :

Theorem 2.4 *Suppose that Ω is a bounded regular domain , and that F, h, and V satisfy the previous assumptions. Suppose that $\lambda < \bar{\lambda}$ and that u satisfies*

$$\begin{cases} F(x, \nabla u, D^2 u) + h(x) \cdot \nabla u |\nabla u|^\alpha + (V(x) + \lambda)|u|^\alpha u \geq 0 & \text{in } \Omega \\ u \leq 0 & \text{on } \partial\Omega. \end{cases}$$

Then $u \leq 0$ in Ω.

We now recall the following comparison principle which holds without assumptions on the regularity of the bounded domain Ω,

Proposition 2.5 *Suppose that $\beta(x, .)$ is non decreasing and $\beta(x, 0) = 0$, that w is an upper semicontinuous sub-solution of*

$$F(x, \nabla w, D^2 w) + h(x) \cdot \nabla w |\nabla w|^\alpha - \beta(x, w(x)) \geq g$$

and u is a lower semicontinuous supersolution of

$$F(x, \nabla u, D^2 u) + h(x) \cdot \nabla u |\nabla u|^\alpha - \beta(x, u(x)) \leq f$$

with g lower semicontinuous, f upper semicontinuous , $f < g$ in Ω and

$$\limsup(w(x_j) - u(x_j)) \leq 0,$$

for all $x_j \to \partial\Omega$. Then $w \leq u$ in Ω.

Remark 2.6 *The result still holds if β is increasing and $f \leq g$ in Ω.*

The proof is as in [2].
We also recall the following weak comparison principle.

Theorem 2.7 *Suppose that Ω is some bounded open set. Suppose that F satisfies (H1), (H2), and (H4), that h satisfies (H5) and V is continuous and bounded. Suppose that $f \leq 0$, f is upper semi-continuous and g is lower semi-continuous with $f \leq g$.*

Suppose that there exist u and v continuous , $v \geq 0$, satisfying

$$\begin{aligned} F(x, \nabla u, D^2 u) + h(x) \cdot \nabla u |\nabla u|^\alpha + V(x)|u|^\alpha u &\geq g \quad in \quad \Omega \\ F(x, \nabla v, D^2 v) + h(x) \cdot \nabla v |\nabla v|^\alpha + V(x))v^{1+\alpha} &\leq f \quad in \quad \Omega \\ u &\leq v \quad\quad on \quad \partial\Omega. \end{aligned}$$

Then $u \leq v$ in $\overline{\Omega}$ in each of these two cases:
1) If $v > 0$ on $\overline{\Omega}$ and either $f < 0$ in Ω, or $g(\bar{x}) > 0$ on every point $\bar{x}$ such that $f(\bar{x}) = 0$,
2) If $v > 0$ in Ω, $f < 0$ in $\overline{\Omega}$ and $f < g$ on $\overline{\Omega}$.

3 Barriers in non smooth domains

In this section we assume that Ω satisfies the exterior cone condition. More precisely we assume that there exists $\bar{r}$ and $\psi \in]0, \pi[$ such that for each $z \in \partial\Omega$ the set $\Omega \cap B(z, \bar{r})$ is included in the open cone which, up to change of coordinates can be given by

$$T_\psi = \{r \in]0, \bar{r}[, 0 \leq \arccos\left(\frac{x_N}{r}\right) \leq \psi\}$$

choosing the main direction of the cone to be e_N. Indeed, in that case, the exterior of Ω contains at least the set of (x', x_N) with $-1 \leq \frac{x_N}{r} \leq \cos\psi$, $r < \bar{r}$.

On the operator F we suppose that it satisfies conditions (H1), (H2), (H3) and (H4), while h satisfies (H5).

3.1 Local barriers

Under the exterior cone condition we are going to construct a local barrier i.e. for any $z \in \partial\Omega$, a supersolution in a neighborhood of z, of $F[v] + h(x) \cdot \nabla v |\nabla v|^\alpha \leq -1$, such that $c|x - z|^\gamma \leq v(x) \leq C|x - z|^\gamma$ for some $\gamma \in (0, 1]$ and for some constant c and C which depend on $\psi, a, A, \gamma, \bar{r}$. This barrier is constructed on the model of those given by Miller for the Pucci operators in [15].

We define
$$v = |x - z|^\gamma \varphi(\theta)$$

where $\theta = \arccos\left(\frac{x_N - z_N}{|x - z|}\right)$. Without loss of generality, we suppose that $z = 0$.

We suppose first that $h \equiv 0$ and at the end of the proof, we will say which are the changes that need to be done when $h \not\equiv 0$. We shall first show that there exists φ a solution of some differential linear equation such that v is a super solution of
$$F(x, \nabla v, D^2 v) \leq -b$$

where b is a positive constant that depends only on ψ, γ, r_o and the structural constant of the operator. It will be useful for the following to observe that $1 \geq \frac{x_N}{r} \geq \cos\psi$ on the considered set.

Let $x = (x_1, \cdots, x_N) = (x', x_N)$. Let $r = |x|$ and $r' = |x'|$. We shall also use the following notation $X' = (x', 0)$.

One has:
$$\nabla v = \gamma r^{\gamma-2} x \varphi(\theta) + r^\gamma \varphi'(\theta) \nabla\theta$$

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and

$$D^2 v \;=\; r^{\gamma-2}\varphi\gamma \left(I + \frac{(\gamma-2)}{r^2} x \otimes x \right)$$
$$+ r^{\gamma-2}\varphi' \left(r^2 D^2\theta + \gamma(\nabla\theta \otimes x + x \otimes \nabla\theta) \right)$$
$$+ r^{\gamma-2}\varphi'' \left(r^2 \nabla\theta \otimes \nabla\theta \right).$$

We now suppose that $\varphi \geq 0$, $\varphi' \leq 0$ and $\varphi'' \leq 0$ then

$$\mathcal{M}^+_{a,A}(D^2 v) \;\leq\; r^{\gamma-2} \left(\varphi\gamma \mathcal{M}^+_{a,A}\left(I + \frac{(\gamma-2)}{r^2} x \otimes x \right) + \right.$$
$$+ \; \varphi' \mathcal{M}^-_{a,A}(r^2 D^2\theta + \gamma(\nabla\theta \otimes x + x \otimes \nabla\theta))$$
$$+ \; \varphi'' \mathcal{M}^-_{a,A}(r^2 \nabla\theta \otimes \nabla\theta)).$$

Since we need to find the eigenvalues of the above matrices let us remark that

$$\nabla\theta = \frac{1}{r'}(\frac{x_N x}{r^2} - e_N) = \frac{x^\perp}{r^2}$$

with

$$x^\perp = \frac{x_N}{r'}X' - r'e_N = \cot\theta x - \frac{r^2}{r'}e_N.$$

In particular $x^\perp \cdot x = 0$ and $|x^\perp| = r$ and we obtain

$$\mathcal{M}^+_{a,A}\left(I + \frac{(\gamma-2)}{r^2} x \otimes x \right) \;=\; A(N-1) + a(\gamma-1)$$
$$\mathcal{M}^-_{a,A}(r^2 \nabla\theta \otimes \nabla\theta) \;=\; ar^2 |\nabla\theta|^2 = a$$
$$\mathcal{M}^-_{a,A}(\gamma(\nabla\theta \otimes x + x \otimes \nabla\theta)) \;=\; \gamma|\nabla\theta|r(a-A) = \gamma(a-A).$$

To complete the calculation we need to compute

$$D^2\theta \;=\; -\frac{1}{r'^2}\frac{X'}{r'} \otimes \left(\frac{x_N x}{r^2} - e_N \right) +$$
$$+ \frac{1}{r'}\left[\frac{1}{r^2}e_N \otimes x - \frac{2x_N}{r^4}x \otimes x + \frac{x_N}{r^2}I \right]$$
$$= \; -\frac{x_N}{(r')^3 r^2}X' \otimes X' +$$
$$+ \frac{1}{r'r^2}\left[X' \otimes e_N + e_N \otimes X' + x_N e_N \otimes e_N - 2\frac{x_N}{r^2}x \otimes x + x_N I \right].$$

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To estimates the eigenvalues of $r^2 D^2 \theta$ we shall use the following facts and notations:

I_{N-1} indicate the identity $(N-1) \times (N-1)$ matrix,

$$I' = \begin{pmatrix} I_{N-1} & 0 \\ 0 & 0 \end{pmatrix}, \quad I = I' + e_N \otimes e_N,$$

$$x \otimes x = X' \otimes X' + x_N(X' \otimes e_N + e_N \otimes X') + x_N^2 e_N \otimes e_N.$$

Then

$$
\begin{aligned}
r^2 D^2 \theta \quad = \quad & \frac{x_N}{r'}\left(-\frac{1}{(r')^2} X' \otimes X' + I'\right) + \frac{x_N}{r'}(2 - 2\frac{x_N^2}{r^2}) e_N \otimes e_N \\
& + \frac{1}{r'}(1 - 2\frac{x_N^2}{r^2})(X' \otimes e_N + e_N \otimes X') - 2\frac{x_N}{r'}\frac{1}{r^2} X' \otimes X'.
\end{aligned}
$$

One has

$$
\mathcal{M}_{a,A}^-\left(\frac{x_N}{r'}\left(-\frac{1}{r'^2} X' \otimes X' + I'\right) + \frac{x_N}{r'}(2 - 2\frac{x_N^2}{r^2}) e_N \otimes e_N\right) \geq -A\frac{x_N^-}{r'}(N-1)
$$
$$
\geq -A(N-1)(\cot\psi)^-
$$

and using $\frac{|2x_N r'|}{r^2} \leq 1$

$$
\mathcal{M}_{a,A}^-(-2\frac{x_N}{r'}\frac{1}{r^2} X' \otimes X') \geq -2\frac{A|x_N|r'}{r^2} \geq -A.
$$

From this one gets that

$$
\begin{aligned}
\mathcal{M}_{a,A}^-(r^2 D^2 \theta) \quad \geq \quad & -A\left((N-1)(\cot\psi)^- + 1\right) \\
& + \mathcal{M}_{a,A}^-(\frac{1}{r'}(1 - 2\frac{x_N^2}{r^2})(X' \otimes e_N + e_N \otimes X') \\
\geq \quad & -|1 - 2\frac{x_N^2}{r^2}|A - A\left((N-1)(\cot\psi)^- + 1\right) \\
\geq \quad & -A - A\left((N-1)(\cot\psi)^- + 1\right) \\
\geq \quad & -A\left((N-1)(\cot\psi)^- + 2\right)
\end{aligned}
$$

where we have used that $|1 - 2\frac{x_N^2}{r^2}| \leq 1$.

Putting everything together we have obtained:

$$\begin{aligned}
\mathcal{M}_{a,A}^+(D^2v) \;&\leq\; r^{\gamma-2}\left(\varphi\gamma(A(N-1)+a(\gamma-1))\right.\\
&\qquad\left.-\varphi'\left(A(N-1)(\cot\psi)^- + 2\right)+\gamma(A-a)\right)+a\varphi'')\\
&\leq\; r^{\gamma-2}\left(\varphi\gamma(A(N-1)+a(\gamma-1))\right.\\
&\qquad\left.-\varphi'\left(A((N-1)(\cot\psi)^- + 2)+\gamma(A-a)\right)+a\varphi''\right).
\end{aligned}$$

With $\beta = A((N-1)(\cot\psi)^- + 2)+\gamma(A-a)$ we shall choose φ such that

$$a\varphi'' - \beta\varphi' + \varphi\gamma A(N-1) = 0$$

and such that for θ in some interval $[0,\psi]$:

$$\varphi > 0, \quad \varphi' \leq 0, \quad \varphi'' \leq 0.$$

Indeed, the solutions are given by

$$\varphi = C_1 e^{\sigma_1\theta} + C_2 e^{\sigma_2\theta}$$

with σ_1 and σ_2 being the positive constants $\sigma_1 = \frac{1}{2}\left(\beta + \sqrt{\beta^2 - 4\gamma\left(\frac{(N-1)A}{a}\right)}\right)$

$\sigma_2 = \frac{1}{2}\left(\beta - \sqrt{\beta^2 - 4\gamma\left(\frac{(N-1)A}{a}\right)}\right)$ where γ is sufficiently close to zero in

order that $\beta^2 > 4\gamma\left(\frac{(N-1)A}{a}\right)$. In that case σ_1 and σ_2 are both positive, one also has $\sigma_1 > \sigma_2$. We prove that for γ small enough, one can find a solution φ such that on $[0,\psi]$, $\varphi \geq 1$, $\varphi' \leq 0$ and $\varphi'' \leq 0$.

We choose $C_1 < 0$ and $C_2 > 0$ with

$$\begin{cases} C_1\sigma_1 + C_2\sigma_2 = 0 \\ C_1 e^{\sigma_1\psi} + C_2 e^{\sigma_2\psi} = 1 \end{cases}$$

This system has a solution because for γ small enough

$$\begin{aligned}
e^{(\sigma_2-\sigma_1)\psi} \;&\geq\; e^{-\beta\psi} \geq \frac{4\gamma(N-1)A}{a\beta^2}\\
&\geq\; 1 - \sqrt{1 - \frac{4\gamma(N-1)A}{a\beta^2}}\\
&\geq\; \frac{\beta - \sqrt{\beta^2 - \frac{4\gamma(N-1)A}{a}}}{\beta}
\end{aligned}$$

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$$\geq \frac{\beta - \sqrt{\beta^2 - \frac{4\gamma(N-1)A}{a}}}{\beta + \sqrt{\beta^2 - \frac{4\gamma(N-1)A}{a}}}$$

$$= \frac{\sigma_2}{\sigma_1}.$$

We now deduce from this that $\varphi' \leq 0$, and $\varphi'' \leq 0$ on $[0, \psi]$.

Indeed the assumption implies that $\varphi'(0) = 0$. Then for $\theta > 0$

$$\varphi'(\theta) = C_1 \sigma_1 e^{\sigma_1 \theta} + C_2 \sigma_2 e^{\sigma_2 \theta} \leq (C_1 \sigma_1 + C_2 \sigma_2) e^{\sigma_1 \theta} = 0.$$

One also has

$$\varphi''(0) = C_1 \sigma_1^2 + C_2 \sigma_2^2 = -C_2 \sigma_1 \sigma_2 + C_2 \sigma_2^2 \leq 0$$

and for $\theta > 0$

$$\varphi''(\theta) = C_1 \sigma_1^2 e^{\sigma_1 \theta} + C_2 \sigma_2^2 e^{\sigma_2 \theta} \leq e^{\sigma_1 \theta}(C_1 \sigma_1^2 + C_2 \sigma_2^2).$$

Let us note that

$$1 \leq \varphi(\theta) \leq \varphi(0) =: C_1 + C_2 = C_2(1 - \frac{\sigma_2}{\sigma_1})$$

and

$$|\varphi'(\theta)| \leq |\varphi'(\psi)| = C_2 \sigma_2 (e^{\sigma_1 \psi - \sigma_2 \psi}).$$

Let $C_\psi = \sup(\varphi^2 + (\varphi')^2)^{\frac{\alpha}{2}}$. We have obtained that

$$\begin{aligned}
F(x, \nabla v, D^2 v) &\leq |\nabla v|^\alpha \mathcal{M}_{a,A}^+(D^2 v) \\
&\leq \gamma^\alpha r^{(\gamma-1)\alpha}(\varphi^2 + (\varphi')^2)^{\frac{\alpha}{2}} \mathcal{M}_{a,A}^+(D^2 v) \\
&\leq -a\gamma^{2+\alpha}(1-\gamma)\varphi r^{(\gamma-1)\alpha + \gamma - 2}(\varphi^2 + (\varphi')^2)^{\frac{\alpha}{2}} \\
&\leq -a\gamma^{2+\alpha} C_\psi r^{\gamma(\alpha+1)-\alpha-2}.
\end{aligned}$$

We now consider the case $h \neq 0$. The above computations give

$$\begin{aligned}
F(x, \nabla v, D^2 v) + h(x) \cdot \nabla v |\nabla v|^\alpha &\leq -C_\psi r^{\gamma(\alpha+1)-\alpha-2}\gamma^{2+\alpha} a\varphi \\
&\quad + |h|_\infty (\gamma r^{\gamma-1})^{1+\alpha} \sup(|\varphi|^2 + (\varphi')^2)^{\frac{1+\alpha}{2}} \\
&< -\frac{C_\psi r^{\gamma(\alpha+1)-\alpha-2}\gamma^{2+\alpha} a}{2} \\
&\leq -\frac{C_\psi r_o^{\gamma(\alpha+1)-\alpha-2}\gamma^{2+\alpha} a}{2} \\
&:= -b
\end{aligned}$$

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for $r \leq r_o := \inf(\bar{r}, \frac{\gamma a}{C_\psi^{\frac{1}{\alpha}} |h|_\infty})$. This ends the proof.

Remark 3.1 *In the same manner one can construct a local barrier by below, i.e. some continuous non positive function w_z' such that $w_z'(z) = 0$ which in the cone is a sub-solution of*

$$F[w_z'] + h(x) \cdot \nabla w_z' |\nabla w_z'|^\alpha \geq 1.$$

3.2 Global barriers and existence.

We now construct a global barrier which will allow us to prove the following existence result.

Proposition 3.2 *Let Ω and F satisfy the previous assumptions. Then, there exists u_o a nonnegative viscosity solution of*

$$\begin{cases} F(x, \nabla u_o, D^2 u_o) + h(x) \cdot \nabla u_o |\nabla u_o|^\alpha = -1 & in \quad \Omega \\ u_o = 0 & on \quad \partial\Omega \end{cases} \tag{3.1}$$

which is γ Hölder continuous.

This Proposition 3.2 will be the first step in the proof of the maximum principle and the construction of the principal eigenfunction for non smooth bounded domains.

The global barrier is given in

Proposition 3.3 *For all $z \in \partial\Omega$, there exists a continuous function W_z on $\overline{\Omega}$, such that $W_z(z) = 0$, $W_z > 0$ in $\Omega \setminus \{z\}$ which is a super solution of*

$$F(x, \nabla W_z, D^2 W_z) + h(x) \cdot \nabla W_z |\nabla W_z|^\alpha \leq -1 \quad in \quad \Omega. \tag{3.2}$$

Remark 3.4 *In the same way, using Remark 3.1 one can construct a continuous function W_z' on $\overline{\Omega}$, such that $W_z'(z) = 0$, $W_z' < 0$ in $\Omega \setminus \{z\}$ which is a sub- solution of*

$$F(x, \nabla W_z', D^2 W_z') + h(x) \cdot \nabla W_z' |\nabla W_z'|^\alpha \geq 1 \quad in \quad \Omega. \tag{3.3}$$

Proof: We argue on the model of [8]. Choose any point $y \notin \Omega$ and r_1 such that $2r_1 < d(y, \partial\Omega)$. Let $G_1(x) = \frac{1}{r_1^\sigma} - \frac{1}{|x-y|^\sigma}$ then

$$
\begin{aligned}
F[G_1] + h(x) \cdot \nabla G_1 |\nabla G_1|^\alpha & \leq \sigma^{1+\alpha} |x-y|^{-(\sigma+1)\alpha-\sigma-2} \left(AN - (\sigma+2)a \right. \\
& \qquad \left. + |h|_\infty |x-y| \right) \\
& \leq -(r_1)^{-\sigma(\alpha+1)-\alpha-2} \sigma^{1+\alpha} \frac{AN}{4}
\end{aligned}
$$

as soon as

$$
\sigma + 2 > \sup\left(\frac{4AN}{a}, \frac{2|h|_\infty \operatorname{diam}\Omega}{a}\right).
$$

Moreover

$$
\frac{1}{r_1^\sigma} \geq G_1(x) \geq \frac{2^\sigma - 1}{(2r_1)^\sigma}
$$

on $\overline{\Omega}$. Defining $G = \frac{r_o^\gamma r_1^\sigma}{2} G_1$, one gets that $G \leq \frac{r_o^\gamma}{2}$.

We denote by $w_z(x) = |z - x|^\gamma \varphi(\theta)$ some local barrier associated to the point $z \in \partial\Omega$ as constructed in the previous section. Let

$$
V_z(x) = \min(G(x), w_z).
$$

Since the infimum of two super-solution is a super solution, V_z is a super-solution of

$$
\begin{aligned}
F[V_z] \; + \; & h(x) \cdot \nabla V_z |\nabla V_z|^\alpha \\
\leq \; & -\kappa^{1+\alpha} \\
= \; & \sup\left(-\frac{C_\psi r_o^{\gamma(\alpha+1)-\alpha-2} \gamma^{2+\alpha} a}{2}, -\frac{r_o^\gamma}{2}(r_1)^{-\sigma(\alpha+1)-\alpha-2} \sigma^{1+\alpha} \frac{AN}{4} \right).
\end{aligned}
$$

Multiplying by κ we get that $W_z = \frac{V_z}{\kappa}$ will denote the super-solution of (3.3).

Remark 3.5 *Observe that, since $G > 0$ in $\overline{\Omega}$ there exists δ such that $W_z(x) = \frac{w_z(x)}{\kappa}$ if $|x - z| < \delta$. Furthermore, by the uniform exterior cone condition there exists $C_w > 0$ such that if $|x - z| < \delta$*

$$
W_z(x) \leq C_w |x - z|^\gamma,
$$

where C_w depends on γ, r_o and ψ and is independent of $z \in \partial\Omega$.

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In the next proposition we shall see that existence of global barriers allows to prove Hölder's regularity for solutions in non smooth domains:

Proposition 3.6 *Let H_j be a sequence of bounded open regular sets such that $H_j \subset \overline{H_j} \subset H_{j+1}$, $j \geq 1$, whose union equals to Ω. Let u_j be a sequence of bounded solutions of*

$$\begin{cases} F(x, \nabla u_j, D^2 u_j) + h(x) \cdot \nabla u_j |\nabla u_j|^\alpha = f_j & in \quad H_j \\ u_j = 0 & on \ \partial H_j. \end{cases}$$

with f_j uniformly bounded. Then there exist C and $\gamma > 0$ independent on j such that

$$|u_j(x) - u_j(y)| \leq C|x - y|^\gamma$$

for all $x, y \in \Omega$, where $\gamma \in (0, 1)$ is given in the previous construction.

Proof: Since ∂H_j is $\mathcal{C}^2$, it satisfies the exterior sphere condition and a fortiori the exterior cone condition. Since the H_j converge to Ω which satisfies the exterior cone condition, we can choose exterior cones with opening ψ and height r_o which do not depend on j.

Using the global barriers of Proposition 3.3 and the comparison principle in H_j one easily has that for any $z \in \partial H_j$

$$u_j \leq |f_j|_\infty^{\frac{1}{1+\alpha}} W_z, \quad in \quad H_j.$$

Let

$$\Delta_\delta = \{(x, y) \in H_j^2 \quad \text{such that} \quad |x - y| \leq \delta\}.$$

Let $C = \max\{\frac{2|u|_\infty}{\delta^\gamma}, C_w |f_j|_\infty^{\frac{1}{1+\alpha}}\}$, we want to prove that for δ small enough, and for any $(x, y) \in \Delta_\delta$

$$u_j(x) - u_j(y) \leq C|x - y|^\gamma. \tag{3.4}$$

In the first step we prove it on the boundary of Δ_δ. Indeed if $|x - y| = \delta$ it is immediate from the definition of C. Suppose hence that $x \in H_j$ and $y \in \partial H_j$, with $|x - y| \leq \delta$. Then, using Remark 3.5, for δ sufficiently small

$$u_j(x) \leq |f_j|_\infty^{\frac{1}{1+\alpha}} W_y \leq C_w |f_j|_\infty^{\frac{1}{1+\alpha}} |x - y|^\gamma.$$

The second step is to check that the inequality (3.4) holds inside Δ_δ. It proceeds exactly as in the smooth case (see [11, 3]) using hypothesis (H2) and (H3).

Proof of Proposition 3.2.

Let H_j be a sequence of bounded open regular sets such that $H_j \subset \overline{H_j} \subset H_{j+1}$, $j \geq 1$, with the union equals to Ω.

Let u_j for $j \geq 1$ be the solution of

$$\begin{cases} F(x, \nabla u_j, D^2 u_j) + h(x) \cdot \nabla u_j |\nabla u_j|^\alpha = -1 & \text{in} \quad H_j \\ u_j = 0 & \text{on} \ \partial H_j. \end{cases}$$

Using the global barriers of Proposition 3.3 and the comparison principle in H_j one easily has that

$$u_j \leq W_z \quad \text{in} \quad H_j.$$

As a consequence, $(u_j)_{j \geq 1}$ is a bounded and increasing sequence - in the sense that $u_j \geq u_{j-1}$ on H_{j-1}-. Using Proposition 3.6, the sequence $(u_j)_j$ is uniformly γ-Hölder continuous. As a consequence on any compact set $J \subset \Omega$, one gets that $(u_j)_j$ converges uniformly to some u_o which satisfies

$$F(x, \nabla u_o, D^2 u_o) + h(x) \cdot \nabla u_o |\nabla u_o|^\alpha = -1.$$

Furthermore u_o equals 0 on the boundary since, by passing to the limit in the previous inequality

$$u_o \leq W_z,$$

for all $z \in \partial \Omega$. We have also obtained that u_o is γ Hölder continuous.

Remark 3.7 *In the same manner it is possible to prove that there exists u'_o a non positive γ-Hölder continuous solution of*

$$\begin{cases} F(x, \nabla u'_o, D^2 u'_o) + h(x) \cdot \nabla u'_o |\nabla u'_o|^\alpha = 1 & \text{in} \quad \Omega \\ u'_o = 0 & \text{on} \quad \partial \Omega, \end{cases}$$

with $u'_o \geq W'_z$ for all $z \in \partial \Omega$.

Corollary 3.8 *Given $f \in \mathcal{C}(\overline{\Omega})$ there exists u a γ-Hölder continuous viscosity solution of*

$$\begin{cases} F(x, \nabla u, D^2 u) + h(x) \cdot \nabla u |\nabla u|^\alpha = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial \Omega \end{cases} \tag{3.5}$$

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with

$$|u(x)| \leq |f|_\infty^{\frac{1}{1+\alpha}} \sup(u_o(x), -u'_o(x)).$$

Furthermore if $f \leq 0$, $u \geq 0$.

Proof Let z_j be a sequence of solutions on H_j of

$$\begin{cases} F(x, \nabla z_j, D^2 z_j) + h(x) \cdot \nabla z_j |\nabla z_j|^\alpha = f & \text{in} \quad H_j, \\ z_j = 0 & \text{on} \quad \partial H_j. \end{cases}$$

By the comparison principle on H_j, $u'_o|f|_\infty^{\frac{1}{1+\alpha}} \leq z_j \leq u_o|f|_\infty^{\frac{1}{1+\alpha}}$. Using Proposition 3.6, the sequence (z_j) is uniformly γ Hölder continuous and then z_j converges on every compact set in Ω to a solution z which is γ Hölder continuous.

If $f \leq 0$, each z_j is non-negative, which implies that $z \geq 0$. Using the inequality

$$|z_j|_\infty \leq |f|_\infty^{\frac{1}{1+\alpha}} u_o$$

in H_j, one gets the final inequality by passing to the limit .

Remark 3.9 *Observe that the existence of u_o and z solutions of (3.1) and (3.6) can be done via Perron's method adapted to viscosity solutions. In particular choosing*

$$u = \sup\{v, \text{ subsolution of (3.6) satisfying}, |f|_\infty^{\frac{1}{1+\alpha}} (W')^\star \leq v \leq |f|_\infty^{\frac{1}{1+\alpha}} W_\star\}$$

where $W_\star$ is the lower semi-continuous envelope of $\inf_{z \in \partial\Omega} W_z$ and $(W')^\star$ is the upper semi continuous envelope of $\sup_{z \in \partial\Omega} W'_z$ (The definition of viscosity solution is then intended in the sense of semi-continuous viscosity solutions, see [3]).

Remark 3.10 *For $V(x) \leq 0$, u_o is a supersolution of*

$$F(x, \nabla u_o, D^2 u_o) + h(x) \cdot \nabla u_o |\nabla u_o|^\alpha + V(x) u_o |u_o|^\alpha = -1.$$

This implies that for any $f \leq 0$ there exists u solution of

$$\begin{cases} F(x, \nabla u, D^2 u) + h(x) \cdot \nabla u |\nabla u|^\alpha + V(x) u |u|^\alpha = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{3.6}$$

and

$$u(x) \leq |f|_\infty^{\frac{1}{1+\alpha}} u_o(x).$$

Proposition 3.11 *The function u_o in Proposition 3.2 satisfies also : $\forall \delta$, there exists K, a compact set in Ω such that*

$$\sup_{\Omega \setminus \overline{K}} |u_o| \leq \delta.$$

Proof. For each $z \in \partial \Omega$ we know that

$$u_o \leq W_z, \quad \text{in} \quad \Omega.$$

Let $\delta > 0$ then for all $z \in \partial \Omega$ there exists r_z such that for $x \in B(z, r_z) \cap \Omega$

$$W_z(x) \leq \delta.$$

Since $\partial \Omega$ is compact one can extract from $\cup B(z, r_z)$ a finite recovering, say $\cup_{i \leq k} B(z_i, r_{z_i})$. Let then K be a compact set such that

$$\Omega \setminus K \subset \cup_{i \leq k} B(z_i, r_{z_i}).$$

We have

$$u_o \leq W = \inf_{z_i, i \leq k} W_{z_i},$$

and then $u_o \leq \delta$ in $\Omega \setminus K$. This ends the proof.

Corollary 3.12 $\forall M > 0$, *there exists K compact subset of Ω, large enough, such that*

$$\bar{\lambda}(\Omega \setminus K) > M.$$

Proof. Let δ be such that $\left(\frac{1}{\delta} \right)^{1+\alpha} \geq M + |V|_\infty$, and let K be large enough in order that

$$\sup_{\Omega \setminus K} |u_o| \leq \delta.$$

Then

$$F[u_o] + h(x) \cdot \nabla u_o |\nabla u_o|^\alpha + (M + V(x)) u_o^{1+\alpha} = -1 + (M + V(x)) u_o^{1+\alpha} \leq 0$$

in $\Omega \setminus K$, and since u_o is positive one gets that $\bar{\lambda}(\Omega \setminus K) \geq M$.

3.3 Maximum principle

Definition 3.13 *We shall say that $\limsup_{x \to \partial\Omega} w(x) \leq 0$ if for all $\epsilon > 0$ there exists K compact in Ω, large enough in order that $\sup_{\Omega \setminus K} w \leq \epsilon$*

Proposition 3.14 *Let $\beta(x, \cdot)$ be a nondecreasing continuous function such that $\beta(x, 0) = 0$. Suppose that w is uppersemicontinuous and bounded by above and satisfies*

$$F(x, \nabla w, D^2 w) + h(x) \cdot \nabla w |\nabla w|^\alpha - \beta(x, w) \geq 0$$

with

$$\limsup w(x_j) \leq 0$$

for all $x_j \to \partial\Omega$. Then $w \leq 0$ in Ω.

Remark 3.15 *If β is increasing then the result holds without requiring any regularity on the bounded domain Ω. In that case one can use comparison principle in Proposition 2.5.*

Proof:

We assume by contradiction that $w > 0$ somewhere in Ω. Let $\bar{x}$ be a point in Ω such that $w(\bar{x}) > 0$, and let $\gamma > 0$ be such that $\gamma u_o(\bar{x}) < w(\bar{x})$. The function

$$w - \gamma u_o$$

is uppersemicontinuous, bounded by above and it admits a supremum > 0, achieved inside Ω. Indeed, let $\epsilon < \frac{w(\bar{x}) - \gamma u_o(\bar{x})}{2}$. Let K be compact and large enough, in order that $\bar{x} \in K$ and such that $w(x) \leq \epsilon$ in $\Omega \setminus K$. Then $(w - \gamma u_o)(x) \leq \epsilon$ in $\Omega \setminus K$. As a consequence $w - \gamma u_o$ achieves its maximum inside K. The end of the proof is the same as in the case of regular sets :

We introduce $\psi_j(x, y) = w(x) - \gamma u_o(y) - \frac{j}{q}|x - y|^q$. One can prove as in [2], that for j large enough, ψ_j achieves its maximum on (x_j, y_j) inside $\Omega \times \Omega$, (more precisely in $K \times K$), and that there exists (X_j, Y_j) in $\mathcal{S}^2$ such that

$$(j|x_j - y_j|^{q-2}(x_j - y_j), X_j) \in J^{2,+} w(x_j)$$

$$(j|x_j - y_j|^{q-2}(x_j - y_j), -Y_j) \in J^{2,-} \gamma u_o(y_j)$$

Moreover one can choose $x_j \neq y_j$ for j large enough, as it is done in [2].

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One has then using (H2), (H4) and the decreasing properties of β,

$$
\begin{aligned}
0 \;\leq\; & F(x_j, j|x_j - y_j|^{q-2}(x_j - y_j), X_j) + h(x_j) \cdot |x_j - y_j|^{(q-1)(\alpha+1)-1}(x_j - y_j) \\
& -\beta(x_j, w(x_j)) \\
\leq\; & F(y_j, j|x_j - y_j|^{q-2}(x_j - y_j), -Y_j) + h(y_j) \cdot |x_j - y_j|^{(q-1)(\alpha+1)-1}(x_j - y_j) \\
& +o(1) \\
\leq\; & -\gamma^{1+\alpha} + o(1)
\end{aligned}
$$

a contradiction since $\gamma > 0$.

4 Existence of an eigenfunction

We recall that V is some bounded and continuous function and that $\bar{\lambda}(\Omega)$ is defined as :

$$
\bar{\lambda}(\Omega) = \sup\{\lambda, \exists\ \varphi > 0 \text{ in } \Omega, F[\varphi] + h(x) \cdot \nabla\varphi|\nabla\varphi|^\alpha + (V(x) + \lambda)\varphi^{1+\alpha} \leq 0\}.
$$

Theorem 4.1 *Let Ω be a bounded domain which satisfies the uniform exterior cone condition, F satisfies condition (H1) to (H4) and h satisfies (H5). There exists a positive function ϕ solution of*

$$
\begin{cases}
F(x, \nabla\phi, D^2\phi) + h(x) \cdot \nabla\phi|\nabla\phi|^\alpha + (V(x) + \overline{\lambda}(\Omega))\phi^{1+\alpha} = 0 & in \quad \Omega \\
\phi = 0 & on \quad \partial\Omega,
\end{cases}
$$

which is γ-Hölder continuous.

Proof :

Let H_j be a sequence of regular subsets of Ω, strictly increasing, with union Ω. One has for $\mu_j = \bar{\lambda}(H_j)$ the existence of $\phi_j > 0$ an eigenfunction in H_j. One also assume that $\sup \phi_j = 1$. Let $\mu = \lim \mu_j \geq \bar{\lambda}(\Omega)$. (Let us note that the sequence (μ_j) is decreasing).

Since the ϕ_j are uniformly bounded, we can apply Proposition 3.6 with $f_j = (V(x) + \mu_j)\phi_j^{1+\alpha}$ and we obtain that the sequence $(\phi_j)_j$ is uniformly Hölder continuous. Up to a subsequence, the sequence (ϕ_j) converges to ϕ a non-negative solution of

$$
F(x, \nabla\phi, D^2\phi) + h(x) \cdot \nabla\phi|\nabla\phi|^\alpha + (V(x) + \mu)\phi^{1+\alpha} = 0.
$$

We have to prove that ϕ is not identically zero.

Let K_1 be a compact set of Ω, such that $\bar{\lambda}(\Omega \setminus K_1) > \mu_1 = \bar{\lambda}(H_1) > \bar{\lambda}(\Omega)$, this is possible according to Corollary 3.12.

Let δ be small enough in order that

$$(\bar{\lambda}(\Omega \setminus K_1) + |V|_\infty)\delta^{1+\alpha} < 1.$$

According to Proposition 3.11, there exists K_2, a compact regular set, such that $K_1 \subset K_2$ and $\sup_{\Omega \setminus K_2} u_o < \delta$.

One has also $\bar{\lambda}(H_j \setminus K_2) \geq \bar{\lambda}(\Omega \setminus K_1) > \bar{\lambda}(\Omega)$.

We observe that u_o satisfies in $\Omega \setminus K_2$, (hence also in $H_j \setminus K_2$) :

$$F(x, \nabla u_o, D^2 u_o) + h(x) \cdot \nabla u_o |\nabla u_o|^\alpha + (\bar{\lambda}(\Omega \setminus K_1) + |V|_\infty)u_o^{1+\alpha} \leq 0$$

which implies in particular that

$$F(x, \nabla u_o, D^2 u_o) + h(x) \cdot \nabla u_o |\nabla u_o|^\alpha + (\bar{\lambda}(H_j) + V)u_o^{1+\alpha} \leq 0.$$

On $\partial(H_j \setminus K_2)$

$$\phi_j \leq 1 \leq \frac{1}{\inf_{K_2} u_o} u_o,$$

hence using the comparison principle Theorem 2.7 on the set $H_j \setminus K_2$, one gets that

$$\phi_j \leq \frac{1}{\inf_{K_2} u_o} u_o \qquad (4.1)$$

on $H_j \setminus K_2$.

Let K_3 which contains K_2 such that in $\Omega \setminus K_3$, $u_o \leq \frac{\inf_{K_2} u_o}{2}$, then

$$\phi_j \leq \frac{1}{2} \quad \text{in} \quad \Omega \setminus K_3.$$

This implies that $\sup_{K_3} \phi_j = 1$, and hence $\sup_{K_3} \phi = 1$.

In particular we have obtained that ϕ is not zero and by the strict maximum principle $\phi > 0$ in Ω. Furthermore $\mu \leq \bar{\lambda}(\Omega)$ and hence $\mu = \bar{\lambda}(\Omega)$.

Passing to the limit in (4.1) we also get that ϕ is zero on the boundary.

5   Other maximum principle and eigenvalues

In all the results of this section we still assume that Ω has the uniform exterior cone condition and F and h satisfy (H1) to (H5).

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We define

$$\lambda_e = \sup\{\bar{\lambda}(\Omega'), \ \Omega \subset\subset \Omega', \ \Omega' \text{ regular and bounded}\}$$

and

$$\tilde{\lambda} = \sup\{\lambda, \exists \ \varphi > 0 \text{ in } \overline{\Omega}, F[\varphi] + h(x) \cdot \nabla\varphi|\nabla\varphi|^\alpha + (V(x) + \lambda)\varphi^{1+\alpha} \leq 0\}.$$

Let us note that the existence of u_o implies that $\tilde{\lambda} > 0$. On the other hand $\lambda_e \geq \bar{\lambda} > 0$.

In this section we are going to prove that $\lambda_e = \tilde{\lambda}$ and that it is an "eigenvalue" in the sense that there exists some $\phi_e > 0$, which satisfies

$$\begin{cases} F(x, \nabla\phi_e, D^2\phi_e) + h(x) \cdot \nabla\phi_e|\nabla\phi_e|^\alpha + (V(x) + \lambda_e)\phi_e^{1+\alpha} = 0 & \text{in} \quad \Omega \\ \phi_e = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Observe that clearly $\lambda_e \leq \bar{\lambda}$, and furthermore if Ω is smooth, the equality holds. The case where Ω is non smooth is open and the identity of these values is equivalent to the existence of a maximum principle for $\lambda < \bar{\lambda}$.

Let us start with the following maximum principle

Proposition 5.1 *For $\lambda < \tilde{\lambda}$, if w is a sub solution of*

$$F(x, \nabla w, D^2 w) + h(x) \cdot \nabla w|\nabla w|^\alpha + (V(x) + \lambda)w^{1+\alpha} \geq 0$$

satisfying

$$w(x) \leq 0 \quad on \quad \partial\Omega$$

then $w \leq 0$ in Ω.

Sketch of the proof : Let $\varphi > 0$ on $\bar{\Omega}$, such that

$$F[\varphi] + h(x) \cdot \nabla\varphi|\nabla\varphi|^\alpha + (V(x) + \lambda)\varphi^{1+\alpha} \leq 0.$$

Suppose that $w > 0$ somewhere, since $\varphi > 0$ on $\overline{\Omega}$ one can define $\gamma' = \sup_{x\in\Omega} \frac{w}{\varphi}$ and follow the proof of [4] to derive a contradiction.

Remark 5.2 *Observe that for $\lambda < \bar{\lambda}$, we don't know if the maximum principle holds when Ω is not smooth because for supersolutions satisfying $\varphi > 0$ in Ω we don't know if $\sup_{x\in\Omega} \frac{w}{\varphi}$ is bounded.*

Proposition 5.3 *There exists $\phi_e > 0$ which satisfies*

$$\begin{cases} F[\phi_e] + h(x) \cdot \nabla\phi_e|\nabla\phi_e|^\alpha + (V(x) + \lambda_e)\phi_e^{1+\alpha} = 0 & in \quad \Omega \\ \phi_e = 0 & on \quad \partial\Omega. \end{cases}$$

Proof of Proposition 5.3.

Let Ω_j be a decreasing sequence of regular open bounded domains which contain Ω. Let ϕ_j be some positive eigenfunction for Ω_j such that $|\phi_j|_\infty = 1$, which exists according to the results in [4].

Using the comparison principle Proposition 2.5, one has that for all $z \in \partial\Omega_j$

$$\phi_j \leq (|V|_\infty + \bar\lambda(\Omega_j))W_z^j$$

where W_z^j is a global barrier for Ω_j. As in Remark 3.5, W_z^j satisfies

$$W_z^j(x) \leq C|x - z|^\gamma$$

with C independent of j and z, since the Ω_j converge to Ω which satisfies the uniform exterior sphere condition. This implies that for $\epsilon > 0$ there exists K compact in Ω, large enough in order that

$$\sup_j \sup_{\Omega_j \setminus K} \phi_j \leq \epsilon.$$

In particular ϕ_j has the property that if $d(K, \partial\Omega_j) < \left(\frac{\epsilon}{C}\right)^{\frac{1}{\gamma}}$.

$$\sup_{\Omega_j \setminus K} \phi_j(x) \leq \epsilon$$

Let K be a compact set in Ω such that $d(K, \partial\Omega) < \epsilon$. Since the distance is continuous, for j large enough $d(K, \partial\Omega_j) < \epsilon$ and then

$$\sup_{\Omega_j \setminus K} \phi_j(x) \leq \epsilon.$$

In particular one can take a compact K large enough in Ω in order that

$$\sup_{\Omega_j \setminus K} \phi_j \leq \frac{1}{2}$$

and then the supremum of ϕ_j is achieved in K.

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By the uniform estimates in Proposition 3.6 the sequence $(\phi_j)_j$ is uniformly γ-Hölder on K and one can then extract from $(\phi_j)_j$ a subsequence such that ϕ_j converges to some function ϕ_e which is such that $|\phi_e|_{L^\infty(K)} = 1$. By compacity one has that ϕ_e is a solution of

$$F(x, \nabla\phi_e, D^2\phi_e) + h(x) \cdot \nabla\phi_e |\nabla\phi_e|^\alpha + (V(x) + \lambda_e(\Omega))\phi_e^{1+\alpha} = 0 \text{ in } \Omega.$$

Moreover $\phi_e > 0$ in Ω, and the estimate

$$\phi_j \le C \inf_{z\in\partial\Omega_j} W_z^j$$

gives, by passing to the limit, that $\phi_e = 0$ on the boundary of Ω.

Corollary 5.4
$$\lambda_e = \tilde{\lambda}$$

Proof: Suppose by contradiction that $\lambda_e < \tilde{\lambda}$, then by the maximum principle one would obtain that $\phi_e \le 0$.

We now present some existence result for the Dirichlet problem.

Proposition 5.5 *Let $\lambda < \lambda_e$ then for any function $f \le 0$ and continuous there exists u a viscosity solution of*

$$\begin{cases} F(x, \nabla u, D^2 u) + h(x) \cdot \nabla u |\nabla u|^\alpha + (V(x) + \lambda)u^{1+\alpha} = f & in \quad \Omega \\ u = 0 & on \quad \partial\Omega. \end{cases} \quad (5.1)$$

Furthermore $u \ge 0$ and γ-Hölder continuous.

Proof. For $K = 2|V|_\infty + |\lambda|$ let u_n be the sequence of solutions of

$$\begin{cases} F[u_{n+1}] + h(x) \cdot \nabla u_{n+1} |\nabla u_{n+1}|^\alpha + (V(x) + \lambda - K)u_{n+1}^{1+\alpha} = f - Ku_n^{\alpha+1} & in \quad \Omega \\ u_{n+1} = 0 & on \quad \partial\Omega \end{cases}$$

with $u_1 = 0$, u_n exists by Remark 3.10. The sequence $(u_n)_n$ is increasing by the comparison principle in Proposition 2.5. Arguing as in [3] one can prove that the sequence is bounded, using the maximum principle of Proposition 3.14. Furthermore there exists a constant C such that

$$u_n \le Cu_o.$$

Passing to the limit, which we can do thanks to the Hölder's regularity given in Propostion 3.6, we get the required solution.

Remark 5.6 *The validity of the maximum principle for $\lambda < \overline{\lambda}(\Omega)$ is equivalent to $\lambda_e = \overline{\lambda}(\Omega)$ and to the existence of a solution for the Dirichlet problem (5.1) for any $\lambda < \overline{\lambda}(\Omega)$.*

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